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# Superconducting $K$ strings in high density QCD

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Abstract: Recently it has been argued that the ground state of high density QCD is likely be a combination of the CFL-phase along with condensation of the $K^0$ field. This state spontaneously breaks a global $\mathrm{U}(1)_Y$ symmetry, therefore one would expect the formation of $\mathrm{U}(1)_Y$ global strings. We discuss the core structure of these strings and demonstrate that under some conditions the global $\mathrm{U}(1)_Y$ symmetry may not be restored inside the string, in contrast with the standard expectations. Instead, $K^+$ condensation occurs inside the core of the string if a relevant parameter $\cos\theta_{K^0} \equiv m_{K^0}^2/\mu_{\mathrm{eff}}^2$ is larger than some critical value $\theta_{K^0} \geq \theta_{\mathrm{crit}}$. If this phenomenon happens, the $\mathrm{U}(1)_Y$ strings become superconducting and may considerably influence the magnetic properties of dense quark matter, in particular in neutron stars.

Keywords: Nonperturbative Effects, QCD, Chiral Lagrangians.



 

## Contents



## 1. Introduction

It is generally believed that there are no topological defects within the Standard Model in vacuum. However, it has been realized quite recently [1]–[5] that some topological defects such as domain walls and strings or vortices may exist within the Standard Model in an unusual environment such as at large baryon density.

In the last few years there has been a renewed interest in high density QCD. Similar to the BCS pairing in conventional superconductivity, the ground state of QCD at high density is unstable due to the formation of a diquark condensate [6, 7, 8] (see [9] for a review). In this new ground state various symmetries, which were present at zero baryon density, are spontaneously broken. Specifically, we discuss in this paper the three flavor $N_f = 3$ color flavor locking (CFL) phase. In this case both the exact $\mathrm{U}(1)_B$ and the approximate axial $\mathrm{U}(1)_A$ symmetries of QCD are spontaneously broken, leading to the formation of $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_B$ global strings described in [3]. The main global property of the strings, the string tension $\alpha$, has been calculated in [3] with the result $\alpha \sim 2\pi v_\pi^2 f_\pi^2 \ln R$, where $f_\pi \sim \mu$ is the corresponding decay constant in dense QCD set by the chemical potential $\mu$; $v_\pi^2 \simeq 1/3$ is the velocity of Goldstone modes in this media, and $R$ is an upper cutoff determined by the environment of the string (for example the presence of other strings).

If, in addition to the CFL phase, kaon condensation also occurs as argued in [10]–[13] then the hypercharge symmetry $\mathrm{U}(1)_Y$ is also spontaneously broken. If this is indeed the case, one more type of strings related to the spontaneously broken $\mathrm{U}(1)_Y$ global symmetry is possible as was discussed in [3]. The string tension $\alpha$ in this case behaves similarly to $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_B$ global strings, and it is determined by the corresponding decay constant $f_\pi \sim \mu$. The next step in studying of $\mathrm{U}(1)_Y$ strings was undertaken in [4] where it was demonstrated that the internal core structure of the $\mathrm{U}(1)_Y$ string could be very different from the $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_B$ global strings described earlier [3]. Namely, it was argued that



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the relevant symmetry may not be restored in the core: in most known cases, in particular in magnetic vortices in a conventional superconductor, the U(1) symmetry is restored inside the core. If this is the case, the $\mathrm{U}(1)_Y$ string becomes superconducting with the core having a $K^+$ condensate. The fact that such unusual behavior, in principle, may occur in the theory of superconducting cosmic strings and in quantum field theory in general, has been known for quite a while [14, 15, 16]. Even more than that, such a behavior has been observed experimentally when the laboratory experiments on $^3He$ provided us with strong evidence for defect core transition in the interior of vortices which appear in the superfluid $^3He-B$ phase (see [17] for a review). Still, such a behavior of the vortex core is considered as an exception rather than a common phenomenon in physics.

Due to the fact that the $\mathrm{U}(1)_Y$ strings might be phenomenologically relevant objects realized in nature (presumably in neutron stars), and due to the property of superconductivity of these strings which might be relevant for the dynamics, we analyze the core structure of $\mathrm{U}(1)_Y$ strings in detail in this letter. More specifically, the goal of this letter is twofold: 1) understand the phenomenon of core transition in the interior of vortices qualitatively, using some analytical methods; 2) make quantitative estimates for phenomenologically relevant parameters in the CFL phase when the transition does occur.

The fact that there should be some kind of phase transition in the string core as a function of the external parameters can be understood from the following simple arguments. If the quark mass difference $m_d - m_u$ is relatively large, than there would be only a U(1) (rather than $\mathrm{SU}(2)_I$) symmetry broken. The standard topological arguments suggest that in this case, the $K^0$ string would be a topologically stable configuration with the restoration of the corresponding U(1) symmetry inside the core (which is a typical situation). If $m_d - m_u$ is exactly zero, such that the isotopical $\mathrm{SU}(2)_I$ symmetry is exact, then symmetry arguments suggest that both $K^0$ and $K^+$ fields condense, and no global stable strings are possible. From these two limiting cases, it is clear that there should be some intermediate region that somehow interpolates (as a function of $m_d - m_u$) between these two cases. Indeed, as we discuss below in detail, the way how this interpolation works is the following. For relatively large $m_d - m_u$ nothing unusual happens: the $K^0$ string has a typical behavior with $K^0$ condensation outside the core, and with restoration of the symmetry inside the core. At some finite magnitude of $m_d - m_u$, an instability arises through the condensation of $K^+$-field inside of the core of the string. As the magnitude of $m_d - m_u$ decreases, the size of the core becomes larger and larger with nonzero values of both $K^0$ and $K^+$ condensates inside the core. Finally, at $m_d = m_u$ the core of the string (with nonzero condensates $K^0$ and $K^+$) fills the entire space, in which case the meaning of the string is completely lost, and we are left with the situation when $\mathrm{SU}(2)_I$ symmetry is exact: no stable strings are possible.

Given this argument, we would expect that there must be some transition region where the $\mathrm{SU}(2)_I$ symmetry is broken to some degree below which $K^+$-condensation will occur inside the core. The point at which this occurs will be estimated in this letter. We will show that $K^+$-condensation only occurs above a certain point $\theta_{K^0} > \theta_{\mathrm{crit}}$, where parametrically $\theta_{\mathrm{crit}}$ is given by $\sin(\theta_{\mathrm{crit}}/2) \simeq$ constant $(\Delta/m_s)\sqrt{(m_d - m_u)/m_s}$, with $\Delta \sim 100\,\mathrm{MeV}$ being the superconducting gap and $m_s$ is the strange quark mass. As we already mentioned, the phenomenon of $K^+$ condensation inside the core of $K^0$ strings in the CFL+$K^0$ phase was discussed earlier in [4]. However, in that paper only a numerical result for some specific parameters was given without analytical calculations showing the region where this phenomenon takes place. We fill this gap and give an explanation of the phenomenon by calculating $\theta_{\rm crit}$ analytically (expressed in terms of the external parameters). Therefore, we can describe the region in parameter space where this phenomenon certainly happens.

This paper is organized as follows. In section 2 we will give a brief overview of the properties of the mixed CFL-$K^0$ phase of high density QCD. Section 3 will discuss the issue of the stability of global $K^0$ strings as a function of the parameter $(m_d - m_u)$. In this section we will calculate $\theta_{\rm crit}$ where $K^+$ condensation occurs inside of the global strings. We end with concluding remarks.

## 2. The CFL+$K^0$ phase of high density QCD

It is well known that the ground state of $N_f = 3$, $N_c = 3$ QCD exhibits the Cooper pairing phenomenon as in conventional superconductivity [7, 8, 18, 19]. The corresponding condensates in the CFL phase take the approximate form:

$$\begin{aligned}\langle q^{ia}_{L\alpha} q^{jb}_{L\beta}\rangle^* &\sim \epsilon_{\alpha\beta\gamma}\epsilon^{ij}\epsilon^{abc}X^{\gamma}_{c}\,,\\ \langle q^{ia}_{R\alpha} q^{jb}_{R\beta}\rangle^* &\sim \epsilon_{\alpha\beta\gamma}\epsilon^{ij}\epsilon^{abc}Y^{\gamma}_{c}\,,\end{aligned} \tag{2.1}$$

where $L$ and $R$ represent left and right handed quarks, $\alpha$, $\beta$, and $\gamma$ are the flavor indices, i and j are spinor indices, a, b, and c are color indices, and $X^{\gamma}_c$ and $Y^{\gamma}_c$ are complex color-flavor matrices describing the Goldstone bosons. In order to describe the light degrees of freedom in a gauge invariant way, one introduces the color singlet field $\Sigma$

$$\Sigma^{\beta}_{\gamma} = XY^{\dagger} = \sum_c X^{\beta}_c Y^{c*}_{\gamma}\,, \tag{2.2}$$

such that the leading terms of the effective lagrangian in terms of $\Sigma$ take the form [11]

$$\begin{aligned}\mathcal{L}_{\rm eff} &= \frac{f_\pi^2}{4}{\rm Tr}\left[\nabla_0\Sigma\nabla_0\Sigma^{\dagger} - v_\pi^2\partial_i\Sigma\partial_i\Sigma^{\dagger}\right] + 2A\left[\det(M){\rm Tr}(M^{-1}\Sigma + {\rm h.c.}\right]\,, \qquad (2.3)\\ \nabla_0\Sigma &= \partial_0\Sigma + i\left(\frac{MM^{\dagger}}{2p_f}\right)\Sigma - i\Sigma\left(\frac{M^{\dagger}M}{2p_f}\right)\end{aligned}$$

where the matrix $\Sigma = \exp(i\pi^a\lambda^a/f_\pi)$ describes the octet of Goldstone bosons with the SU(3) generators $\lambda^a$ normalized as ${\rm Tr}(\lambda^a\lambda^b) = 2\delta^{ab}$. The quark mass matrix in eq. (2.3) is given by $M = {\rm diag}(m_u, m_d, m_s)$. Finally, we neglect the electromagnetic interactions in the expression (2.3) but keep the isospin violation $\sim (m_d - m_u)$, which is an appropriate approximation for the physically relevant case when the baryon density is not very high [13]. The constants $f_\pi, v_\pi$ and $A$ have been calculated in the leading perturbative approximation and are given by [20, 21]:

$$f_\pi^2 = \frac{21 - 8\ln 2}{18}\frac{\mu^2}{2\pi^2}\,, \qquad v_\pi^2 = \frac{1}{3}\,, \qquad A = \frac{3\Delta^2}{4\pi^2}\,. \tag{2.4}$$

Recently, it was realized in [11, 13, 10, 12] that the ground state of the theory may be different from the pure CFL phase for a physical value of the strange quark ($m_s \gg m_u, m_d$): condensation of the $K^0$ and $K^+$ mesons would lower the free energy of the system by reducing the strange quark content. Specifically, it has been argued that kaon condensation would occur in the CFL phase if $m_s \geq 60\,\mathrm{MeV}$. This means that $\Sigma = \mathbf{1}$ is no longer the global minimum of the free energy; instead, some rotated value of $\Sigma$ describes the ground state in this case. In what follows we consider the realistic case when the isospin symmetry is not exact, $m_d > m_u$, such that $K^0$ condensation occurs. The appropriate expression for $\Sigma$ describing the $K^0$ condensed ground state in this case can be parameterized as:

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{K^0} & \sin\theta_{K^0} e^{-i\phi} \\ 0 & -\sin\theta_{K^0} e^{i\phi} & \cos\theta_{K^0} \end{pmatrix}, \tag{2.5}$$

where $\phi$ describes the corresponding Goldstone mode and $\theta_{K^0}$ describes the strength of the kaon condensation with [11]:

$$\cos\theta_{K^0} = \frac{m_0^2}{\mu_{\mathrm{eff}}^2}, \qquad \mu_{\mathrm{eff}} \equiv \frac{m_s^2}{2p_F}, \qquad m_0^2 \equiv a m_u (m_d + m_s), \qquad a = \frac{4A}{f_\pi^2} = \frac{3\Delta^2}{\pi^2 f_\pi^2}. \tag{2.6}$$

In order for this to be satisfied, we must have $m_0 < \mu_{\mathrm{eff}}$. If kaon condensation occurs and $\theta_{K^0} \neq 0$, an additional U(1) symmetry is broken. This brings us to the next section where will discuss the consequences of this symmetry breaking.

## 3. Global $K$-strings

We follow our logic outlined in the Introduction and first consider global $K^0$ strings when they are topologically stable. This corresponds to the approximation when the splitting between $K^0$ and all other degrees of freedom is relatively large such that we can neglect in our effective description all fields except $K^0$. After that we analyze situation when the $K^0$ and $K^+$ masses are degenerate such that the $K$ strings become unstable. Finally we introduce a small explicit isospin violation into our description $\sim (m_d - m_u)$ in order to analyze the stability/instability issue for this physically relevant case. The important global characteristic of the $K^0$ string, the string tension $\alpha$, with logarithmic accuracy is determined by the pion decay constant $\alpha \sim f_\pi^2$ as discussed in [3], and it is not sensitive to the internal structure of the core. The subject of this paper is the analysis of the core structure of the $K^0$ strings.

### 3.1 Topologically stable U(1) strings in $CFL + K^0$ phase

We start by considering the following effective field theory which describes a single complex $K^0$ field. This corresponds to the case of spontaneously broken U(1) symmetry

$$\mathcal{L}_{\mathrm{eff}}(K^0) = |(\partial_0 + i\mu_{\mathrm{eff}})K^0|^2 - v_\pi^2 |\partial_i K^0|^2 - m_0^2 |K^0|^2 - \lambda |K^0|^4 . \tag{3.1}$$

We fix all parameters of this effective theory by comparing the amplitude of the $K^0$ field to the result obtained from the theory described in the previous section. We neglect all other degrees of freedom at this point (see the discussion at the end of this section). If $\mu_{\rm eff} > m_0$ the kaon field acquires a nonzero vacuum expectation value $\langle K^0 \rangle = \eta/\sqrt{2}$ where

$$\eta^2 = \frac{\mu_{\rm eff}^2 - m_0^2}{\lambda} = \frac{\mu_{\rm eff}^2}{\lambda}\,(1 - \cos\theta_{K^0})\,. \tag{3.2}$$

For $\mu_{\rm eff} > m_{K^0}$ it is more convenient to represent the effective lagrangian in the familiar form of a Mexican hat type potential:

$$\mathcal{L}_{\rm eff} = |\partial_0 K^0|^2 - v_\pi^2 |\partial_i K^0|^2 - \lambda\left(|K^0|^2 - \frac{\eta^2}{2}\right)^2 . \tag{3.3}$$

This is a text-book lagrangian with spontaneously broken global U(1) symmetry which admits topologically stable global string solutions. As is well-known, global strings are solutions of the time independent equation of motion. The time independent equation of motion for $K^0$ is given by:

$$v_\pi^2 \nabla^2 K^0 = 2\lambda\left(|K^0|^2 - \frac{\eta^2}{2}\right) K^0\,. \tag{3.4}$$

For the $K^0$ string solution we will make the following ansatz:

$$K^0_{\rm string} = \frac{\eta}{\sqrt{2}} f(r) e^{in\phi}\,, \tag{3.5}$$

where $n$ is the winding number of the string (we will take $n = 1$ in what follows), $\phi$ is the azimuthal angle in cylindrical coordinates, and $f(r)$ is a yet to be determined solution of eq. (3.4) which obeys the boundary conditions $f(0) = 0$ and $f(\infty) = 1$. Substituting this ansatz into eq. (3.4), we arrive at the following ordinary differential equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df(r)}{dr}\right) - \frac{f(r)}{r^2} = \frac{\mu_{\rm eff}^2(1-\cos\theta_{K^0})}{v_\pi^2}(f(r)^2 - 1) f(r)\,, \tag{3.6}$$

where we replace $\lambda\eta^2 \to \mu_{\rm eff}^2(1-\cos\theta_{K^0})$ according to eq. (3.2). Although a numerical solution of this equation is possible, the most important part of this section is an analytical analysis and, therefore, we prefer to take a variational approach. We follow [15, 22] and assume a solution of the form:

$$f(r) = 1 - e^{-\beta r}\,, \tag{3.7}$$

with the variational parameter $\beta$. Minimizing the energy with respect to $\beta$, the result is [22]:

$$\beta_{\rm min}^2 = \frac{89}{144}\frac{\mu_{\rm eff}^2}{v_\pi^2}(1-\cos\theta_{K^0})\,. \tag{3.8}$$

This result was compared with the exact numerical solution of (3.6) and is a reasonable approximation. From this equation we see that a typical string core radius is parametrically given by $r_{\rm core} \sim 1/\beta_{\rm min} \sim v_\pi/\mu_{\rm eff}(\sin\theta_{K^0}/2)$,and it becomes smaller when $\mu_{\rm eff}$ gets larger. The first important lesson of this simple exercise is the observation that the global string is stable and the U(1) symmetry is restored inside of the core, $f(r=0) = 0$, as expected. The second important lesson is the observation that the core size becomes large when $\theta_{K^0} \to 0$.

### 3.2 Unstable $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y \to \mathrm{U}(1)$ strings in $CFL + K^0$ phase

Our next task is an analysis of the $K$ string when isospin is an exact symmetry (i.e. $m_u = m_d$), such that the symmetry pattern breaking is $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y \to \mathrm{U}(1)$. In this case, based on topological arguments, we know that the $K$ string is unstable. We want to analyze the stability issue in detail to understand this phenomenon on the quantitative level.

To simplify things we start with the effective lagrangian which only includes a single complex kaon doublet $\Phi = (K^+, K^0)$. As discussed in [11], this is an appropriate approach to discuss kaon condensation in the CFL phase if all other degrees of freedom are much heavier. As before (see eqs. (3.1), (3.3)) we can represent the effective lagrangian in the form of a Mexican hat type potential,

$$\mathcal{L}_{\text{eff}}(\Phi) = |\partial_o \Phi|^2 - v_\pi^2 |\partial_i \Phi|^2 - \lambda \left( |\Phi|^2 - \frac{\eta^2}{2} \right)^2 . \tag{3.9}$$

All parameters here are defined in the same way as in (2.6), (3.2), (3.3) and the $\Phi$ field acquires a non-zero vacuum expectation value which we assume takes the form

$$\langle \Phi \rangle = \left( 0, \frac{\eta}{\sqrt{2}} \right) . \tag{3.10}$$

The time independent equations of motion for $K^0$ and $K^+$ are given by:

$$v_\pi^2 \nabla^2 K^+ = 2\lambda \left( |K^+|^2 + |K^0|^2 - \frac{\eta^2}{2} \right) K^+ , \tag{3.11}$$

$$v_\pi^2 \nabla^2 K^0 = 2\lambda \left( |K^+|^2 + |K^0|^2 - \frac{\eta^2}{2} \right) K^0 . \tag{3.12}$$

For the $K^0$ string solution we will make the following ansatz:

$$K^0_{\text{string}} = \frac{\eta}{\sqrt{2}} f(r) e^{in\phi} , \qquad K^+ = 0 , \tag{3.13}$$

such that $f(r)$ is the solution of eq. (3.6) with the boundary conditions $f(0) = 0$ and $f(\infty) = 1$. Without calculations, from topological arguments we know that although the solution (3.13) satisfies the equation of motion (3.11), (3.12), it is an unstable solution. The source of instability can be seen as follows. We follow the standard procedure and expand the energy in the $K^0$ string background to quadratic order in $K^+$ and $K^0$ modes:

$$E(K^0 = K^0_{\text{string}} + \delta K^0,\, K^+) \approx E_{\text{string}} + \delta E . \tag{3.14}$$

We know that the $K^0$ string itself is a stable configuration, therefore, the $\delta K^0$ modes cannot have negative eigenvalues which would correspond to the instability. Therefore, we concentrate only on the "dangerous" modes related to $K^+$ fluctuations, in which case $\delta E$ is given by:

$$\delta E = \int d^2 r \left( v_\pi^2 |\nabla K^+|^2 + \mu_{\text{eff}}^2 (1 - \cos\theta_{K^0})(f(r)^2 - 1)|K^+|^2 \right) , \tag{3.15}$$

where $f(r)$ is the solution of eq. (3.6) with the boundary conditions $f(0) = 0$ and $f(\infty) = 1$.



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If $\delta E$ is a positive quantity, then the $K^0$ string is absolutely stable and $K^+$ modes do not destroy the string configuration. If $\delta E$ is negative, this means that this is a direction in configuration space where the $K^0$ string decays. Following [23], the $K^+$ field can be expanded in Fourier modes:

$$K^+ = \frac{\eta}{\sqrt{2}} \sum_m g_m(r) e^{im\phi} \,. \tag{3.16}$$

Now we have the $K^+$ field in terms of the dimensionless Fourier components $g_m(r)$. Setting $m = 0$ in the above expansion in order to analyze the lowest energy $\delta E_0$ contribution in (3.15), we arrive at:

$$\delta E_0 = \frac{\eta^2}{2} \int d^2 r \left( v_\pi^2 \left( \frac{\partial g_0(r)}{\partial r} \right)^2 + \mu_{\rm eff}^2 (1 - \cos\theta_{K^0}) \cdot (f^2(r) - 1) g_0^2(r) \right) . \tag{3.17}$$

In order to have dimensionless coordinates and fields, we will perform the following change of variables, $\tilde{r} = \gamma r$, where $\gamma \equiv \mu_{\rm eff}/v_\pi$. This change of variables sets the string width in $f(\tilde{r})$ to be $\tilde{r}_{\rm core} \sim 1$. Equation (3.17) now reads:

$$\delta E = \frac{\eta^2 v_\pi^2}{2} \int d^2\tilde{r}\, g_0(\tilde{r}) \hat{O} g_0(\tilde{r}) \,, \quad \hat{O} = -\frac{1}{\tilde{r}} \frac{d}{d\tilde{r}} \left( \tilde{r} \frac{d}{d\tilde{r}} \right) + \lambda' (f^2(\tilde{r}) - 1) \,, \quad \lambda' = (1 - \cos\theta_{K^0}) \,. \tag{3.18}$$

The problem is reduced to the analysis of the two-dimensional Schrodinger equation for a particle in an attractive potential $V(r) = -(1 - \cos\theta_{K^0})(1 - f^2(\tilde{r}))$ with $f(\tilde{r})$ being the solution of eq. (3.6) with the boundary conditions $f(0) = 0$ and $f(\infty) = 1$. Such a potential is negative everywhere and approaches zero at infinity. As is known from standard quantum mechanics [24], for an arbitrarily weak potential well there is always a negative energy bound state in one and two spatial dimensions; in three dimensions a negative energy bound state may not exist. For the two dimensional case (the relevant problem in our case) the lowest energy level of the bound state is always negative and exponentially small for small $\lambda'$. One should note that our specific potential $V(r) = -(1 - \cos\theta_{K^0})(1 - f^2(\tilde{r}))$ which enters (3.18) is not literally the potential well, however one can always construct the potential well $V'$ such that its absolute value is smaller than $|V(r)|$ everywhere, i.e. $|V'| < |V(r)|$ for all $r$. For the potential well $V'$ we know that the negative energy bound state always exists; when $V'$ is replaced by $V$ it makes the energy eigenvalue even lower. Therefore, the operator (3.18) has always a negative mode irrespective of the local properties of function $f(r)$. As a consequence, the string (3.5) is an unstable solution of the classical equation of motion, the result we expected from the beginning from topological arguments. The instability manifests itself in the form of a negative energy bound state solution of the corresponding two-dimensional Schrodinger equation (3.18) irrespective of the magnitudes of the parameters.

### 3.3 $K^+$-condensation in the core of $K^0$ strings in $CFL + K^0$ phase

The issue of the stability or instability of $K^0$ strings reviewed in the previous section is highly sensitive to the degree of symmetry present in the lagrangian describing the $K^0/K^+$ system. If the SU$(2)_I$ symmetry is strongly broken, the $K^0$ strings will be absolutely (topologically) stable as discussed in subsection 3.1. If the $SU(2)_I$ symmetry remains unbroken, the $K^0$ strings will always be unstable as discussed in subsection 3.2. Now we introduce an explicit symmetry breaking parameter $\delta m^2$ into (3.9) fixed by the original lagrangian (2.3) such that our simplified version of the system (only $\Phi = (K^+, K^0)$ fields are taken into account) has the form[1]

$$\mathcal{L}_{\text{eff}}(\Phi) = |\partial_0 \Phi|^2 - v_\pi^2 |\partial_i \Phi|^2 - \lambda \left( |\Phi|^2 - \frac{\eta^2}{2} \right)^2 - \delta m^2 \Phi^\dagger \tau_3 \Phi \,, \qquad \delta m^2 \equiv \frac{a}{2} m_s (m_d - m_u) \,. \tag{3.19}$$

We anticipate that, as the symmetry breaking parameter $\delta m^2$ in eq. (3.19) becomes sufficiently large, a stable $K^0$ string with restored U(1) symmetry in the core, $f(r = 0) = 0$, must be reproduced, as discussed in subsection 3.1. When the symmetry breaking parameter $\delta m^2$ in becomes sufficiently small, one should eventually reach a point where $K^+$ instability occurs, and it is energetically favorable for a $K^+$ condensate to be formed inside the core of the string.

In this subsection we calculate the critical value of $\theta_{K^0}$ when the $K^+$ instability occurs, and a $K^+$ condensate does form in the string core. In addition to this, we will obtain an estimate of the absolute value of the $K^+$-condensate at the center of the core of the string ($r = 0$). In order to determine if $K^+$-condensation occurs within the core of $K^0$-strings, we would like (ideally) to solve the set of coupled differential equations

$$v_\pi^2 \nabla^2 K^+ = 2\lambda \left( |K^+|^2 + |K^0|^2 - \frac{\eta^2}{2} \right) K^+ + \delta m^2 K^+ \,, \tag{3.20}$$

$$v_\pi^2 \nabla^2 K^0 = 2\lambda \left( |K^+|^2 + |K^0|^2 - \frac{\eta^2}{2} \right) K^0 - \delta m^2 K^0 \,. \tag{3.21}$$

with the appropriate boundary conditions. This is not a trivial task, and we will follow a different approach. The main point is: we are mainly interested in the *critical values* of the parameters when $K^+$-condensation starts to occur inside the core. In this case we can treat $K^+$ field as a small perturbation in the $K^0$ background field. Such an approach is not appropriate when $K^+$-condensation is already well-developed in which case both fields $K^0$ and $K^+$ must be treated on the same footing. However, this approach is quite appropriate when one studies the transition from the phase where $K^+$ background field is zero to the region where it becomes nonzero.

To begin, we will expand the energy in the constant $K^0$ string background to quadratic order in $K^+$:

$$E \approx E_{\text{string}} + \delta E \,, \tag{3.22}$$

[1] In addition to the mass splitting proportional to the difference $m_d - m_u$, there is also a splitting due to electromagnetic effects, $\delta m_{EM}^2 \sim \alpha_{EM} \Delta^2/(8\pi)$. As mentioned in section 2, the electromagnetic contribution becomes important at very large chemical potential [13]. However, this correction can be neglected for the present work since we are not considering large chemical potentials and it only amounts to a 10% correction, $\delta m_{EM}/\delta m \sim 0.1$. In order to remain self-consistent, we neglect all electromagnetic contributions throughout this paper.



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where $\delta E$ is given by:

$$\delta E = \int d^2r \left( v_\pi^2 |\nabla K^+|^2 + \mu_{\text{eff}}^2 (1 - \cos\theta_{K^0}) \cdot (f^2(r) - 1)|K^+|^2 + \frac{a}{2} m_s(m_d - m_u)\,|K^+|^2 \right). \tag{3.23}$$

If $\delta E$ is a positive quantity, then the $K^0$ string is stable and $K^+$ condensation does not occur inside the core of the string. If $\delta E$ is negative, this means that it is energetically favorable for $K^+$ condensation to occur inside the core of the string. We follow the same procedure as before keeping the most "dangerous" mode to arrive at:

$$\delta E = \frac{\eta^2}{2} \int d^2r \left( v_\pi^2 \left( \frac{\partial g_0(r)}{\partial r} \right)^2 + \mu_{\text{eff}}^2 (1 - \cos\theta_{K^0}) \cdot \left(f^2(r) - 1\right) g_0^2(r) + \right.$$
$$\left. + \frac{a}{2} m_s(m_d - m_u) g_0^2(r) \right). \tag{3.24}$$

In dimensionless variables this expression can be represented as follows

$$\delta E = \frac{\eta^2 v_\pi^2}{2} \int d^2\tilde{r}\, g_0(\tilde{r})[\hat{O} + \epsilon] g_0(\tilde{r}), \tag{3.25}$$

where

$$\hat{O} = -\frac{1}{\tilde{r}} \frac{d}{d\tilde{r}} \left( \tilde{r} \frac{d}{d\tilde{r}} \right) + (1 - \cos\theta_{K^0})(f^2(\tilde{r}) - 1), \qquad \epsilon \equiv \frac{a}{2} \frac{m_s(m_d - m_u)}{\mu_{\text{eff}}^2}. \tag{3.26}$$

The only difference between this expression and eq. (3.18) describing the instability of the string in case of exact isospin symmetry, is the presence of the term $\sim \epsilon$ in eq. (3.25). The problem of determining when $K^+$-condensation occurs is now reduced to solving the Schrodinger type equation $\hat{O} g_0 = \hat{E} g_0$. From the previous discussions we know that $\hat{E}$ for the ground state is always negative. However, to insure the instability with respect to $K^+$-condensation one should require a relatively large negative value i.e. $\hat{E} + \epsilon < 0$. It can not happen for arbitrary weak coupling constant $\sim (1 - \cos\theta_{K^0})$ when $\theta_{K^0}$ is small. However, it does happen for relatively large $\theta_{K^0}$. To calculate the minimal critical value $\theta_{\text{crit}}$ when $K^+$-condensation develops, one should calculate the eigenvalue $\hat{E}$ as a function of parameter $\theta_{K^0}$ and solve the equation $\hat{E}(\theta_{\text{crit}}) + \epsilon = 0$. As we mentioned earlier, for very small coupling constant $\lambda' = (1 - \cos\theta_{K^0}) \to 0$ the bound state energy is negative and exponentially small, $\hat{E} \sim -e^{-1/\lambda'}$. However, for realistic parameters of $\mu$, $\Delta$, $m_s$, $m_u$, $m_d$ the parameter $\epsilon$ is not very small and we expect that in the region relevant for us the bound state energy $\hat{E}$ is the same order of magnitude as the potential energy $\sim \lambda'$. In this case we estimate $\theta_{\text{crit}}$ from the following conditions $-\hat{E}(\theta_{\text{crit}}) \sim \lambda' \sim (1 - \cos\theta_{\text{crit}}) \sim \epsilon$ with the result which can be parametrically represented as

$$\sin\frac{\theta_{\text{crit}}}{2} \sim \text{const} \frac{\Delta}{\pi f_\pi} \sqrt{\frac{m_s(m_d - m_u)}{\mu_{\text{eff}}^2}} \sim \text{const} \frac{\Delta}{m_s} \sqrt{\frac{(m_d - m_u)}{m_s}}, \tag{3.27}$$

where we have neglected all numerical factors in order to explicitly demonstrate the dependence of $\theta_{\text{crit}}$ on the external parameters. The limit of exact isospin symmetry, which corresponds to $m_d \to m_u$ when the string becomes unstable, can be easily understood from the expression (3.27). Indeed, in the case that the critical parameter $\theta_{\rm crit} \to 0$ becomes an arbitrarily small number the $K^+$ instability would develop for arbitrarily small $\theta_{K^0} > 0$. The region occupied by the $K^+$ condensate at this point is determined by the behavior of lowest energy mode $g_0$ at large distances, $g_0(\tilde{r} \to \infty) \sim \exp(-\hat{E}\tilde{r})$ such that a typical $\tilde{r} \sim (m_d - m_u)^{-1} \to \infty$ as expected.

In order to make a quantitative, rather than qualitative estimation of the critical value $\theta_{\rm crit}$ when $\hat{E}(\theta_{\rm crit}) + \epsilon = 0$, we discretize the operator $\hat{O}$ and solve the problem numerically, with the boundary conditions $g_0(\infty) = 0$ and $g_0(0) = $ constant. Varying the condensation angle $\theta_{K^0}$, we see that a negative eigenvalue $\hat{E} + \epsilon < 0$ appears when $\theta_{K^0} > \theta_{\rm crit} \approx 53^o$. For our parameters we use $m_u = 5\,{\rm MeV}$, $m_d = 8\,{\rm MeV}$, $m_s = 150\,{\rm MeV}$, $\mu = 500\,{\rm MeV}$, and $\Delta = 100\,{\rm MeV}$ which gives $\epsilon \equiv a m_s(m_d - m_u)/(2\mu_{\rm eff}^2) \simeq 0.1$. In figure 1 we show a plot of $g_0(\tilde{r} = 0)$ as a function of $\theta_{K^0}$. One can see that the transition from no $K^+$-condensation is reached at about $\theta_{\rm crit} \approx 53^o$. In figure 2 we plot the functions $f(\tilde{r})$ (related to the $K^0$ string by (3.5)) and $g_0(\tilde{r})$ (related to the $K^+$ condensate by (3.16)) as a function of the rescaled coordinate $\tilde{r}$. One can see that the $K^+$ condensate falls off over the same distance as the $K^0$ string reaches its vacuum expectation value. We should note that the solution to the above Schrodinger equation does not give us the overall normalization of the function $g_0(\tilde{r})$. We have estimated the overall normalization of $g_0(\tilde{r})$ by minimizing the total energy of the system using a variational approach.

To conclude the section, we want to make the following comment. We have demonstrated above that $K^+$ condensation might occur in the core of $K^0$ strings if some conditions are met. We also explained how this phenomenon depends on the external parameters. We also derived the equation $\hat{E}(\theta_{\rm crit}) + \epsilon = 0$, the solution of which allows us to calculate the critical parameters when $K^+$ condensation starts to occur. All these discussions were quite general because they were based on the symmetry and topological properties of the system rather than on a specific form of the interaction. However, the numerical estimates presented above was derived by using a concrete form of the effective lagrangian (3.19) describing the lightest $K^+$, $K^0$ degrees of freedom. The question arises how sensitive our numerical results are when the form of the potential changes. To formulate the question in a more specific way, let us remind the reader that, in general, the effective lagrangian describing the Goldstone modes can be represented in many different forms as long as symmetry properties are satisfied. The results for the amplitudes describing the interaction of the Goldstone particles do not depend on a specific representation used. A well-known example of this fact is the possibility of describing the $\pi$ meson properties by using a linear $\sigma$ model as well as a non-linear $\sigma$ model (and many other models which satisfy the relevant symmetry breaking pattern). The results remain the same if one discusses the local properties of the theory (such as $\pi\pi$ scattering length) when the $\pi$ meson is considered as a small quantum fluctuation rather than a large background field . It may not be the case when $\pi$ represents a large background field in which case some numerical difference between different representations of the effective lagrangian may occur. Roughly speaking, the source of the difference is an inequality $\pi(x) \neq \sin\pi(x)$ for large global background fields such as a string solution which is the subject of this letter.



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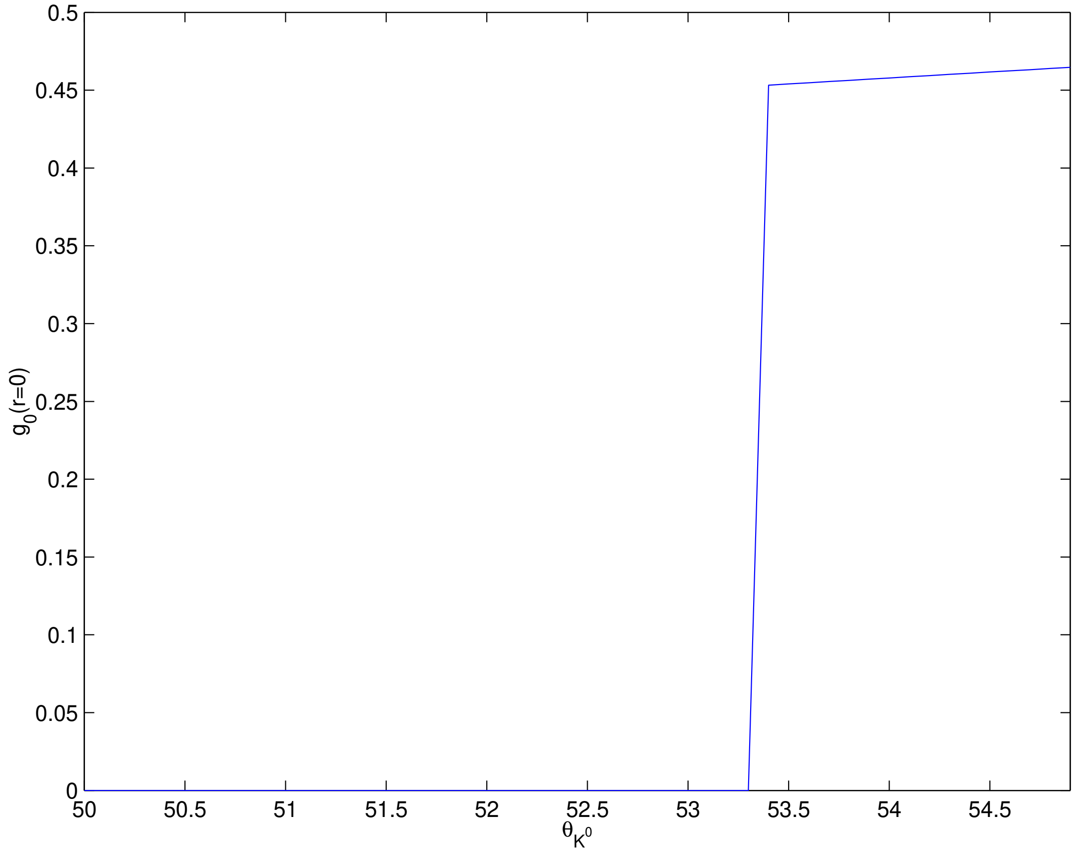


**Figure 1:** In this figure we plot the the value of the $K^+$-condensate at the centre of the $K^0$-string $g_0(r=0)$ ($g_0(r)$ and $K^+$ are related my eq. (3.16)) as a function of the kaon condensation angle $\theta_{K^0}$. From this graph, we can see that $K^+$-condensation occurs at $\theta_{K^0} \gtrsim 53^o$. Note that there is an abrupt point at which $K^+$-condensation occurs inside the core. As described in the text, the transition to the $K^+$-condensed core corresponds to a jump at $\theta_{\rm crit}$. The finite slope on the plot at this point is due to the discretization of the $\theta_{K^0}$ variable.

Having this in mind, we repeated similar numerical estimates discussed above for the original effective lagrangian (2.3) where Goldstone fields represented in the exponential form rather than in form determined by the effective lagrangian (3.19). As before, in these estimates we considered exclusively $K^+$ modes which are the energetically lowest modes and which can potentially destabilize the system presented by the background field of the $K^0$ string. This approximation is justified as long as the $K^+$ field is the lowest massive excitation in the system when a $K^0$ condensate develops. Also, the typical size of the core must be larger than the inverse gap $\Delta^{-1}$, i.e. $r_{\rm core} \simeq v_\pi/\mu_{\rm eff} \gg \Delta^{-1}$ in order to maintain color superconductivity inside the core. We assume this is the case. Our numerical results suggest, that the critical parameters are not very sensitive to the specifics of the lagrangian such that $\theta_{\rm crit}$ is close to our previous numerical estimates. Therefore, most likely, the real world (with our parameters $\theta_{K^0} \simeq 70^o$) case corresponds to $\theta_{K^0} > \theta_{\rm crit}$ and thus, a $K^+$ condensate does develop inside the core of the $K^0$ strings. Therefore, our new result is the description of the region in parameter space where this phenomenon happens.

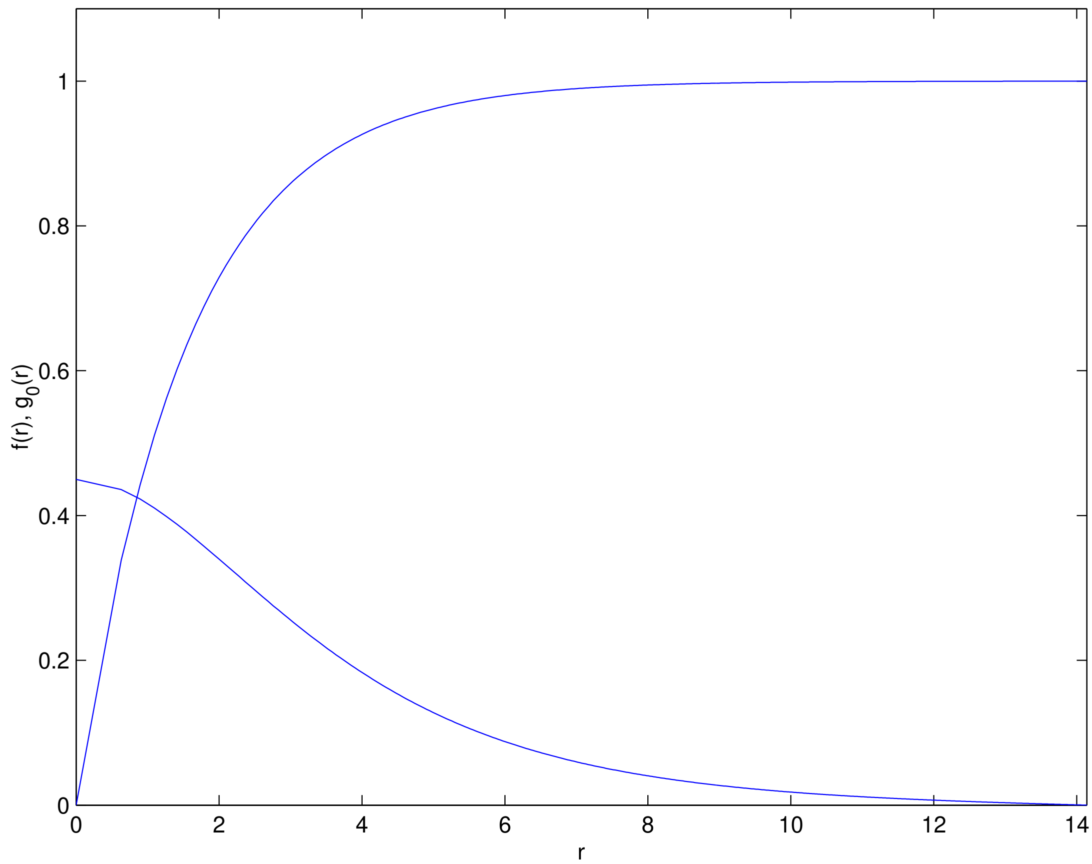


**Figure 2:** In this figure we plot the functions $f(r)$ (related to the $K^0$-string by eq. (3.5)) and $g_0(r)$ (related to the $K^+$-condensate by eq. (3.16)) as a function of the dimensionless rescaled coordinate $\tilde{r} \equiv \mu_{\rm eff} r / v_\pi$.

## 4. Conclusion

We have demonstrated that, within the CFL+$K^0$ phase of QCD, $K^+$ condensation does occur within the core of global $K^0$ strings if some conditions are met: $\theta_{K^0} > \theta_{\rm crit}$. We presented two estimates for $\theta_{\rm crit}$: an analytical one which gives a qualitative explanation of the phenomenon, as well as numerical one for the physically relevant parameters realized in nature. Our results suggest that if a CFL+$K^0$ phase is realized in nature, it is likely that $K^0$ strings form together with $K^+$ condensation inside the core, in which case the strings become superconducting strings (see [14] for a more complete description of superconducting strings and their properties).

It is known that the CFL+$K^0$ phase may be realized in nature in neutron stars interiors and in the violent events associated with collapse of massive stars or collisions of neutron stars, so $K^0$ superconducting strings with $K^+$ condensation inside the core could be very important for such astrophysical phenomena. It has been recently argued [25] that such conditions may also occur in early universe during the QCD phase transition. In this case it might be important for cosmological problems, such as the dark matter problem, as well.

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